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The macrodynamics of an endogenous business cycle model of marxist inspiration

John Cajas Guijarro [a,b,*], Leonardo Vera [c,b]

[a] School of Economic Sciences, Central University of Ecuador, Av. Universitaria, Quito 170129, Ecuador
[b] Department of Development, Environment and Territory, FLACSO-Ecuador, Calle La Pradera E7-174 y Av. Diego de Almagro, Quito, Ecuador
[c] School of Economics, Central University of Venezuela, Ciudad Universitaria, Caracas 1051, Venezuela

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ABSTRACT

This study offers a model that formalizes some of Marx's insights about how capital accumulation generates contradictions that may reproduce never-ending cycles of booms and slumps. The model takes the reserve army of labor as a regulator of the distribution of class-power over the business cycle with a two-sided role: influencing labor productivity, directly through the intensity of labor and indirectly through real wages. The model forms a complex dynamical system capable to yield trajectories for the employment rate, the wage share, and the intensity of labor. Goodwin (1967) model may be considered as a particular case of the model. Complex dynamics may also emerge when we remove some key assumptions and explore and simulate 3-D versions of the system. Though close orbits around non-hyperbolic equilibrium points can be obtained, the possibility of unstable dynamics with increasing amplitudes in the trade cycle and a structural crisis cannot be ruled out.

1. Introduction

Together with Clément Juglar, Karl Marx was one of the earliest thinkers of the XIX century to discern the existence of a rhythmic pattern to business activity and thus extended a line of thought partly initiated by Malthus and Sismondi. In *Das Capital*, in an effort to show the restricted character and the contradictions of capitalist production and the inevitability of recurrent economic crisis, Marx criticized the harmony postulated by Smith and the followers of Ricardo and conceived capitalism as a highly dynamic economic machine, constantly in motion by the profit-motivated behavior of capitalist and the continual and counteracting pressures of capital accumulation, technological change, and the reserve army of unemployed. Although Marx did not work a complete theory of business cycles, except in the most general and fragmented terms, his writings can still be a source of important insights. He clearly saw that counteracting pressures generate crises that automatically lead the system to respond in a cyclical pattern. In his own words: "Effects, in their turn, become causes, and the varying accidents of the whole process, which always reproduces its own conditions, take on the form of periodicity" (Marx [1867], 2010:627).

A theory of the business cycle inspired in Marx as well as its economic implication can be reconstructed from his work if approached with care. Thus, taking as inspiration Marx's insights into the structure and dynamics of production in capitalist economies (that are mostly displayed in Capital), this study offers a modeling structure that formalizes his main reasoning on how a free enterprise economy generates endogenous business cycles. In particular, we are interested in showing how the process of capital accumulation in Marx generates contradictions between relevant forces and tendencies of the economy that in turn reproduce never-ending cycles of booms and slumps that are inextricably related to the cyclical behavior of employment and profitability.

Now, in the past we have witnessed some attempts to formalize Marx's views on periodic crisis within a business cycle framework. However, most of these modern efforts do not entirely rely on Marx's great insights on the structure and dynamics of production in capitalist economies or leave out important aspects closely linked to the macrodynamics contained in Marx. The remarkable work of Sherman (1979, 1991), for instance, is inspired in a synthesis of Marx, Mitchell, Keynes, and Kalecki. Efforts by Eagly (1972) and Laibman (1997, chapter 9), though more "Marxist" in inspiration, pay little or no attention to the distributive cycle and how the succession of stages of animation and crisis are mediated by it as well as by the mechanization and the exploitation of the labor force. In contrast, both studies pay too much attention to the dynamic equilibrium path of supply and demand for

* Corresponding author at: School of Economic Sciences, Central University of Ecuador, Av. Universitaria, Quito 170129, Ecuador.
E-mail address: jcajasg@uce.edu.ec (J. Cajas Guijarro).

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labor and operate in a system in which economic activity manifests itself as an employment cycle.

Undoubtedly, the most known and elegant dynamic formalization of Marx's view of distributive cycles was made more than fifty years ago by Richard Goodwin. The Goodwin class struggle model, which blends aspects of the Harrod–Domar growth model with a real wage Phillips curve, demonstrates that the trend and the cycle are indissolubly fused, and that distributional conflict produces endogenous cycles. In his work, Goodwin (1967) incorporates the Marxian concept of the reserve army of unemployed into a system of two nonlinear differential equations of the Lotka-Volterra type for the employment rate and the wage share as state variables. Goodwin shows that in the course of the dynamic process as the rate of accumulation rises sufficiently, so does the employment ratio, rising worker's bargaining power, the real wage, and the wage share. The process goes on until the rise in the wage share is sufficient to reduce profitability to the point where the accumulation rate slows down and unemployment begins to rise. The increase in the rate of unemployment and the replenishment of the reserve army of labor yields a falling wage share and therefore an upturn in profits and the rate of accumulation.

Several criticism and extensions have been raised around Goodwin's early contribution and over the past decades the model has been expanded in almost all possible directions. This is not the place to address major extensions and contributions. Azevedo et al. (2019) provide an excellent and updated summary in this respect.

Now, though the basic structure of the two-dimensional system (in the wage share and the rate of employment) of the Goodwin model can be a concise statement of the endogenous cycle implied in Capital, that structure leaves out key aspects of the motion of the capitalist economy emphasized in Marx that can make the formal system a bit more complex but substantially richer. For instance, while Goodwin (1967) takes the rate of growth of labor productivity as an exogenous given exponential rate, a most valuable insight of Marx was to recognize that labor productivity is endogenous and that it changes according to changes in labor-saving machinery, real wages as well as by the intensity of labor (in a fight over speed-up by the bosses versus slow-down by the workers). If exploitation is understood as the way the bargaining powers of conflicting classes determine wages, labor intensity, and the working day (Screpanti, 2019), then Goodwin's approach and simplification have very little to say about it.[1] Further, and in contrast with Goodwin (1967) who assumes fixed prices and no reaction from capitalists to variations in the real wage, the interaction of workers and capitalists through wage-price dynamics should be seriously considered.

Then, by introducing the possibility of endogenous labor productivity, the existence of exploitation through changes in labor intensity, and the underlying conflict between labor and capital in the price-wage setting process, this study presents a model that yields a closer version of Marx's view of capitalist control of the work process that results also in a less stylized explanation of a perpetual endogenous cycle.

Our model collapses to a set of differential equations that form an autonomous non-linear and complex system. Using both analytical tools and numerical simulations, the model is capable to yield trajectories to represent the dynamic behavior of the employment rate, the wage share, and the intensity of labor even in a context of (implicit) endogenous inflation.

To make sense of this complex model, some simplified versions are considered. In a slightly simplified version when labor productivity only depends on mechanization and labor intensity and the price of goods remains constant (model A), we prove analytically that the addressed system yields Goodwin's growth cycle model as a particular case, with all the trajectories (in the positive orthant) showing closed orbits around the upper singular point. By removing the fixed price assumption, we obtain a more complex 2D dynamical system still containing the rate of employment and the wage share as state variables (model B). We identify in this 2D system a unique singular point that presents local stability and show that when inflation is considered as an (implicit) endogenous variable, economic stability requires that the parameter that represents the power that capitalists exert over the real wage be below some upper bound working-class target. Notable, such upper bound has an inverse relationship with labor intensity, thus, we show that a high enough power of the capitalist class to reduce the real wage combined with a sufficiently high labor intensity will make the economy more vulnerable to instability and a structural crisis. This type of crisis is different from the endogenous and periodic crisis that emerge during the business cycle and can only be overcome with an exogenous change in the parameters of the model.

By removing the assumption of an exogenous intensity of labor we obtain two further models: A 3D dynamical system where prices are constant and labor intensity is pushed by a set of forces (model C), and a complete version of the general model in which both labor intensity and inflation are endogenous (model D). We apply the Hopf bifurcation theorem for 3D dynamical systems and numerical simulations to analytically evaluate the trajectory and economic stability in these models. In model C we provide an analytical demonstration and numerical simulations to prove the existence of multiple stable limit cycles depending on the initial conditions of the state variables. Those limit cycles emerge when the power of workers to reduce exploitation through labor intensity falls in the neighbourhood of a critical value that seems to generate *supercritical bifurcations*. In model D, when both labor intensity and inflation are endogenous, the bifurcation, in general, leads to stable limit cycles oscillations, but we also show that a sufficiently high power of the capitalist class to reduce the real wage combined with a sufficiently high power of the capitalist class to increase labor intensity will make the economy more vulnerable to instability and a structural crisis. In this sense, we estimate numerically the relationship between these types of power that is necessary for maintaining limit cycles, with the preliminary result that the relationship may be represented by a non-linear function that deserves a deeper discussion when studying the interaction between class-power and crisis.

The rest of the paper is organized as follows. In Section 2 we turn to Marx's insights into the dynamics of production in capitalist economies, paying special attention to the determination of capital accumulation, the rate of profit, and labor productivity, which went through a substantial conceptualization in Capital. Next, we introduce the reserve army of labor, a key regulator of power and exploitation over the business cycle that also drives other variables within the system. The centrality of the reserve army of labor to the Marxist scheme of capitalist dynamics is illustrated through its effects on the costs of labor power as well as on the intensity of labor. Thus, we develop, step by step, the dynamic structure of an endogenous business cycle model of a free enterprise system, whose source of inspiration can be found in Marx's works. Section 3 summarizes our suggested non-linear and complex system and explores analytically and via simulations the qualitative features and properties of four versions of this general system. Finally, some summarizing results and concluding comments are given in Section 4.

2. From capital accumulation to endogenous cycles

2.1. Determinants of capital accumulation

In Capital (particularly, in Volume II) Marx repeatedly uses the concept of the circuit of capital to characterize the dynamics and the structure of the capitalist economy. In this circuit, capital moves through several different forms. If we begin with money capital, (M), then the basic dynamics that rule the capitalist system would be initiated by buying commodities (C) that may be divided into means of production

[1] Here we implicitly follow Veneziani (2013) who argues that from a Marxist point of view a notion of power, or dominance, is necessary to define exploitation.

(MP) and labor power (LP). Thus, when interacting in production (Q), commodities produce new commodities (C'), whose sale generates an amount of money (M') greater than the initial amount spent. The difference between a larger quantity of money (M') and the initial amounted money (M) accounts for profits (P). Such logic reflected in the circuit of capital can be written as follows:

$$M - C\begin{cases} MP \\ LP \end{cases} \ldots Q \ldots C' - M' \text{ where } M' > M \tag{1}$$

When profits emerge, the initial money (M) becomes a representation of total capital (K). From the point of view of the circuit of capital, the first component of the money spent on means of production is termed *constant capital* (c), since in the process of production it does not undergo any quantitative alteration of value.[2] A second component is termed *variable capital* (v), which is somehow a representation of the money spent on acquiring labor power. Variable capital ends up in the hands of the working class that uses it entirely for consumption purposes (Marx [1867], 2010:209–21). This distribution of money allocated to constant and variable capital is captured in expressions (2) to (4).

$$K = c + \rho v \tag{2}$$

$$c = pA \tag{3}$$

$$v = whL \tag{4}$$

where K represents the money capital that should be paid to start production (here $\rho = 1$ if wages are paid when production starts and $\rho = 0$ if wages are paid at the end of production), A is the quantity of means of production employed ;[3] p stands for the price of the means of production (which will be assumed equal to that of all other commodities) ;[4] L is the number of active workers selling their labor power, h is the number of hours of work per worker, and w is the average nominal wage per hour of work.

It is now a simple matter to derive a formula for the rate of profit. Since we are also interested in the accumulation of capital, what matters is the net profits that remain after allowing for depreciation. These we shall normally refer to as profits which are defined as the difference between the gross income obtained after selling a certain volume of goods produced (Q) at a price (p) and the wage bill (v), discounting expenses for the wear or depreciation of the means of production. Note that a fixed proportion δ of constant capital is scrapped each period (where $0 \leq \delta \leq 1$ is a depreciation rate). Such a definition of profits is reflected in (5).

$$\Pi = pQ - (v + \delta c) \tag{5}$$

In turn, the average rate of profit (r) is defined as the ratio of profits to total money capital initially paid, a definition set out in (6).

$$r = \frac{\Pi}{K} \tag{6}$$

Regarding the volume of output produced, this is expressed as the product of the number of people employed, the hours of work, and the average "labor productivity" per hour of work, as indicated in (7).

$$Q = qhL \tag{7}$$

To consider now labor productivity as well as its determinants, let us introduce first the definition of the mechanization of the labor process, which is – according to Marx – a distinguishing historic feature of the capitalist system. Mechanization is understood here as the ratio of the means of production to workers employed,[5] i.e.

$$m = \frac{A}{L} \tag{8}$$

Not only Marx but also Smith believed that mechanization would provide a great boost to industrial productivity. Marx, in particular, even advising that mechanization could have some deleterious effects on employment, observed that modern machinery has "the wonderful power of shortening and fructifying human labor" (Marx [1856], 2010:655).[6]

Changes over time of the means of production with respect to labor can be represented with a permanent increase in mechanization at a constant rate γ_m[7] i.e.

$$\frac{m'}{m} = \gamma_m, \ 0 \leq \gamma_m < 1 \tag{9}$$

where $m' = dm/dt$ represents the change in the mechanization rate in continuous time. Turning then to productivity we will assume, following Marx, that labor productivity (q) increases with increasing mechanization and with increasing labor intensity (ε),[8]as indicated in ((10).[9]

$$q = q(\epsilon, m), \ \frac{\partial q}{\partial \epsilon} > 0, \frac{\partial q}{\partial m} > 0 \tag{10}$$

To generate profits and a positive profit rate ($\Pi > 0$ and $r > 0$), money in the hands of the capitalist class must grow. For that to happen, the *wage share* in the gross income, given by (11), must be much less than 1.

$$\omega = \frac{v}{pQ} = \frac{w}{pq} < 1 \tag{11}$$

More rigorously, since h, L and p are positive, then it can be proved that Π (and r) will be positive if and only if[10]

$$\omega < 1 - \frac{\delta m}{hq} \tag{12}$$

Our next task is to define the accumulation of capital. Capital accumulation for Marx is promoted by competition, which compels

[2] We assume that constant capital is equivalent here to fixed capital. Hence, it does not incorporate any element of circulating capital.

[3] It is assumed that all installed capacity is used.

[4] There is only one good, which can be consumed or used as a mean of production.

[5] Though the Organic Composition of Capital is also the term Marx used to describe the proportion between constant capital and variable capital, this proportion can be viewed from two different but related angles. Firstly, it can be viewed from the perspective of the value of the constant capital as against the value of the variable capital employed. Secondly, it can be viewed from the perspective of the physical amounts of each employed. Marx called the first of these relations the Value Composition of Capital, and the second the Technical Composition of Capital.

[6] Grossman (1992) has correctly pointed out that the displacement of workers by machinery, which Marx describes in Capital (Volume One, Chapter 15, 'Machinery and Modem Industry'), is a technical fact produced by the growth of A relative to L and as such is not a specifically capitalist phenomenon. That machinery replaces labor is an incontrovertible fact that belongs to the very concept of machinery as labour saving means of production.

[7] That the rate of growth of the capital stock per worker is roughly constant over long periods of time is one of Kaldor`s stylized facts that framed the economic growth research agenda.

[8] Intensity of work represents the magnitude of labor power that workers actually use by unit of labor time. When intensity of work increases, there is "a reduction of the 'pores' of labour, i.e. the 'dead', non-utilized segments of time during the work-hour" Mavroudeas and Ioannides 2011:431; Marx [1867] 2010:412–20).

[9] The intensity of work is a category of special importance in the neo-Marxist literature that arises in the mid-1970s and early 1980s to explain the advances or setbacks in labor productivity. In perspective and along this line are the works of Braverman (1974), Marglin (1974) Gintis (1976), Weisskopf, Bowles and Gordon (1983) and Bowles (1985).

[10] When the depreciation rate is zero, the expression is simplified to $\omega < 1$.

individual capitalists to invest and accumulate to survive. Thus, after obtaining profits, to guarantee its existence, the capitalist class would allocate a fraction (or the entire mass) of profits to expand the total capital (purchasing additional amounts of means of production and more labor power), obtain more profits, and expand its power (Marx [1867], 2010:587). Capital accumulation then emerges as an increase in capital that is financed - in the most fundamental case - with a fraction (s) of capitalist profits (Π) less the amount of depreciation (δc) that occurs during the production process.[11] Formally

$$K' = s\Pi - \delta c \tag{13}$$

where $K' = dK/dt$ stands for the change in the stock of capital in continuous time. Our formulation, like Goodwin's model, is classical in the sense that savings determine investment, and accordingly, do not consider the problem of effective demand.[12] Thus, and since the economy is close, saving equals investment and the only use of investment in this economy is to accumulate capital and to cover capital scrapping expenses.

To study the evolution of capital in this economy, we divide K' by K and rewrite the capital accumulation equation in terms of growth. Thus, the rate of growth of capital which is also called the accumulation rate (γ_K) is represented as

$$\gamma_K = \frac{K'}{K} \tag{14}$$

The identification of the deep determinants of the accumulation rate (14) as well as the rate of profit (6) can be quickly sketched in. If we take the rate of capital accumulation and combine it with previous definitions, we obtain

$$\gamma_K = \frac{K'}{K} = \frac{s\Pi - \delta c}{K} = sr - \frac{\delta c}{c + \rho v} = sr - \frac{\delta pA}{pA + \rho whL} = sr - \frac{\delta m}{m + \rho hq\omega} \tag{15}$$

The most relevant intuition from (15) is that the rate of accumulation (γ_K) increases with the rate of profit (r) which is given by taking expression (6) and combining it with previous definitions to get:

$$r = \frac{\Pi}{K} = \frac{pQ - (v + \delta c)}{c + \rho v} = \frac{pqhL - (whL + \delta pA)}{pA + \rho whL} = \frac{hq(1-\omega) - \delta m}{m + \rho hq\omega} \tag{16}$$

Under condition (12), $r > 0$. Inspection of expression (16) shows that the rate of profit increases when labor productivity increases and falls when the wage share increases. Further, it is possible to note that since $q = q(\epsilon, m)$, then through labor productivity the profit rate depends (indirectly) on changes in labor intensity and mechanization.

[11] We assume that workers do not save and there is no circulating (constant) capital.

[12] Though in their diagnosis of the causes of these endogenous fluctuations Keynesian economists would emphasize the importance of autonomous investment decisions as a source of fluctuating demand, a more Marxists approach focus on the effects of class struggle over the distribution of income and saving. Hence, within the Marxian framework that we use the flow of investment is treated then as an accommodating variable which adapts to the flow of saving. We should say however, that the two explanations need not be mutually exclusive and there are some remarkable efforts to develop small and analytically manageable models which embodies Keynesian effective demand problems as well as a Marxian emphasis on class struggle and on the importance of the reserve army of labour. Skott (1989) presents a first effort in this direction combining a Kaldorian business cycle model with a Goodwin model. Further work that describes the economy in terms of interaction between capacity utilization and income distribution can be found in Taylor (2004), Barbosa-Filho and Taylor (2006), Nikiforos and Foley (2012), Taylor, Foley, and Rezai (2018), Dávila-Fernández and Sordi (2019), among others. On the importance to keep aggregate demand in the long-run picture see Marglin (2021).

2.2. On the reserve army of labor and its multiple roles

In Chapter 25 of Capital (Volume One), where Marx derives 'The General Law of Capitalist Accumulation', the layoff of workers through the introduction of machinery is hardly mentioned and explained in detail. Marx limits its analysis to mention the general tendency of machinery to displace labor. From our point of view, what Marx seems to describe in this chapter about capital accumulation in a more sophisticated way is that workers are made redundant not so much because they are displaced by machinery, but because, at a specific level of the accumulation of capital and over the cycle, profits become too small.

How does this fall in profitability happen? With the accumulation of capital, both the constant and variable components of capital can grow (Marx [1867], 2010:608). In the case of variable capital growth ($v' > 0$), if working hours are constant, such growth implies an increase in wages and/or employees. Hence, with the increase in variable capital two effects arise: On the one hand, an increase in wages can motivate workers to increase labor intensity (as we will see later on). As Marx clearly remarked, the accumulation of capital could cause that "the demand for labourers may exceed the supply, and, therefore, wages may rise" (Marx [1867], 2010:609). Close observation of Eq. (10) tells us that if higher wages motivate an increase in labor intensity this, in turn, may generate an improvement in labor productivity. On the other hand, there is the possibility that higher employment increases the power of the working class to pull labor intensity down, negatively affecting productivity. In the end, if the real wage grows more than the "labor productivity", causing an increase in the wage share,[13] ceteris paribus, the rate of profit will fall (see expression 16).

To avoid or mitigate the profit squeeze, capitalism needs some lever that lowers wages and/or increases labor productivity above wage growth (be it with greater mechanization and/or intensity of work). Reactions in this direction harm workers because lower wages subtract subsistence capacity, but also because higher labor productivity achieved with higher intensity or mechanization could diminish the demand for labor and cause layoffs. All this allows us to understand that

> "It is capitalistic accumulation itself that constantly produces, and produces in the direct ratio of its own energy and extent, a relatively redundant population of labourers, i.e., a population of greater extent than suffices for the average needs of the self-expansion of capital, and therefore a surplus population" (Marx [1867], 2010:624).

Thus, the progress of accumulation needs a relative overpopulation: a group of non-employed or underemployed workers[14] that do not reach the social average subsistence. These people are a "relative surplus population" or *reserve army of labor*; replaceable by greater mechanization and/or higher labor intensity. Along with this relative overpopulation, an *active labor army* (a group of employees who reach a subsistence level equal to or even better than the social average) coexists (Marx [1867], 2010:631).

The temporary reduction in the size of the reserve army of labor in comparison to the active labor army at the peak of the business cycle had the effect of pulling up wages above their average value, and profits and the profit rate are correspondingly squeezed. But this affects the accumulation process and then leads to a fall in job creation, higher unemployment, and replenishment of the reserve army. Then, during an economic downturn, many of the workers in the active labor army

[13] From (11) note that $\frac{\omega'}{\omega} = \left(\frac{w'}{w} - \frac{p'}{p}\right) - \frac{q'}{q}$ where the term in parenthesis represents the rate of change of the *real wage*. Thus, $\left(\frac{w'}{w} - \frac{p'}{p}\right) > \frac{q'}{q}$ implies $\frac{\omega'}{\omega} > 0$.

[14] Marx's breakdown of the reserve army of labor into its various components (floating, latent, stagnant, pauperism) was complex indeed. It included not only those who were "wholly unemployed" but also those who were only "partially employed" (see Marx [1867] 2010:634–38).

would themselves be made "redundant," thereby increasing the numbers of unemployed on top of the normal reserve army. In such periods, the enormous weight of the relative surplus population would tend to pull wages down below their average value. It is in this sense that the reserve army becomes the instrument used by capitalism to prevent significant wage increases and thereby maintain profitability. Thus, "independently of the limits of the actual increase of population", the reserve army of labor provides, "for the changing needs of the self-expansion of capital, a mass of human material always ready for exploitation" (Marx [1867], 2010:626).

2.2.1. The Labor reserve function

Introducing the reserve army of labor raises questions regarding its determinants. Marx provides some clues in that respect. According to him:

> "The accumulation of capital, though originally appearing as its quantitative extension only, is affected, as we have seen, under a progressive qualitative change in its composition [both technical and in terms of value], under a constant increase of its constant, at the expense of its variable constituent" (Marx [1867], 2010:623).

Therefore, from this perspective, both the process of capital accumulation and the process of mechanization (represented by the qualitative change in the composition of capital) are interwoven to regulate the course of employment.

Now to derive an analytical version of the function representing the reserve army of labor, it will be assumed that the army only includes unemployed (underemployment does not exist). This is done indirectly by defining an employment rate (l) as the ratio between workers already hired (L) and the total labor force (N), as stated in (17).

$$l = \frac{L}{N} \tag{17}$$

Here it will be assumed that the growth rate of the labor force is exogenous (mainly ruled by demographic aspects) ,[15] which is represented with a constant rate n, as indicated in (18).

$$\frac{N'}{N} = n, \;\; 0 < n < 1 \tag{18}$$

The definition in Eq. (17) is made operational by taking logarithms of the variables and differentiating with respect to time and inserting (18), which gives:

$$\frac{l'}{l} = \frac{L'}{L} - n \tag{19}$$

If in fact the behavior of the rate of growth of the employment rate (l'/l) is affected by the pace of the accumulation process as well as mechanization (as suggested by Marx), then we need to find a way to enter these factors in (19). First, using (2–4), (7–8), and (11), it can be obtained (20).

$$L = \frac{K}{p(m + \rho h q \omega)} \tag{20}$$

Now differentiating (20) with respect to time and substituting (14), we can obtain an expression for the rate of growth of employment, which in turn can be substituted into (19). The resulting relationship is captured in (21).

$$\frac{l'}{l} = (\gamma_k - n) - \pi - \frac{m' + \rho h (q\omega' + q'\omega)}{m + \rho h q \omega} \tag{21}$$

where $\pi = \frac{p'}{p}$ represents the *rate of inflation*. As in Goodwin (1967), the growth of the employment rate is an important endogenous variable that also drives other variables within the system. We call this relationship, represented in (21), the *labor reserve function*. Close inspection of the labor reserve function indicates that the employment rate will increase with an increase in the accumulation rate (that is, when $\gamma_K > 0$), while it will contract with the increase of mechanization (m^{\prime} > 0), when the population available to work increases (that is, when $n > 0$), and when prices increase ($\pi > 0$). In the first case, it can be thought that the employment rate increases thanks to the fact that a greater accumulation increases the demand for labor. On the effect of the hours of work, prices, the wage share, and the rate of inflation on the employment rate, the situation is more complex. Even the growth of the employment rate is highly dependent on the assumption of wages paid at the beginning ($\rho = 1$) or the end ($\rho = 0$) of production.

2.2.2. The Two-sided effect of the reserve army of labor: the wage effect and the intensity effect

The centrality of the reserve army of labor to the Marxist scheme of capitalist dynamics can also be illustrated through its effects on the costs of labor power as well as on the intensity of labor. It is through these links that the reserve army contributes to adjust the exploitation of the working class. In essence, what happens is that throughout the cycle, but in specific during the downturn, workers are forced to offer their labor power for incomes lower than those necessary for the average social subsistence (generating a wage contraction pushed, for instance, by the threat of dismissal). But also in periods of crisis the capitalist class pushes workers to increase their labor intensity as a response to threats of layoff and replacement (Marx [1867], 2010:629) (although without a complete notion of the specific level of that intensity).[16] Thus, we can point out that there are two ways or forces through which an increase in the reserve army increases exploitation: one that reduces wages, workers' motivation, and effort, and another that increases the intensity of work by applying greater surveillance intimidation over workers.

To operationalize these ideas, two expressions are proposed that reflect the pressure of the reserve army to directly affect labor intensity and indirectly thorough a change in real wages. The two expressions we consider are, first an equation for wage dynamics, and second an equation for labor intensity or effort dynamics. Moreover, in reaction to wage dynamics, we add a price adjustment equation.

The dynamics of real wages are determined by two forces; the rate of employment and the rate of inflation. Marx suggested the impact of the rate of employment on wage dynamics by saying:

> "The general movements of wages are exclusively regulated by the expansion and contraction of the industrial reserve army, and these again correspond to the periodic changes of the industrial cycle. They are, therefore, not determined by the variations of the absolute number of the working population, buy by the varying proportions in which the working class is divided into active and reserve army" (Marx [1867], 2010:631).

Real wage dynamics is captured by an inflation-augmented linear version of the real wage Phillips curve presented by Goodwin (1967)

$$\left(\frac{w'}{w} - \pi\right) = -\alpha_{11} + \alpha_{12} l - \alpha_{13}\pi, \;\; 0 < \alpha_{1i} < 1 \tag{22}$$

[15] For the possibility of an endogenous growth of population depending on the dynamics of capital accumulation in a Marxist model, see Harris (1983). For a more recent reference of a model where labour supply indirectly depends on capital accumulation through unemployment, see Marglin (2021, chap.18: 746).

[16] Here we should say, following Bowles (1985), that capitalists cannot cost-lessly know what each worker is doing at any given moment even if they know all of the workers' production capacities and personality characteristics.

where $\frac{w'}{w} - \pi$ represents the rate of change of the real wage (which is equal to $\frac{\omega'}{\omega} + \frac{q'}{q}$, see note 12). Superficially the presence of the term $\alpha_{12}l$ in (22) may resemble the conventional negative relationship between the rate of increase in *real wages* and the percentage of the labor force unemployed. However, the relationship here does not necessarily derive from an excess demand function (as in the conventional approach), but it is linked to the distribution of power (between capitalists and workers) over the evolution of the business cycle. At a lower employment rate (l) (implying a greater relative weight of the reserve army), real wages will tend to decrease, either because the unemployed are willing to offer their labor power at lower wages or because higher unemployment implies a lower organizational capacity of workers, weakening their bargaining power. The relative bargaining power of the working class (or of the union) is also relevant to explain the effects that $\alpha_{13}\pi$ may have on the rate of change of the real wage. Lack of full indexation explains why price inflation π may negatively affect real wages. Thus, this relationship between price and real wage dynamics shows the *power of the capitalist class* to reduce the effective growth of the real wage below the initial target of the working class.

Following Pitchord (1957, 1963) we now consider cost and demand elements as the most convenient method of pricing. Prices react to the higher demand caused by an increase in the employment rate, but also in the context of imperfect competition in the market for goods and facing a wage increase, capitalists might feel able and motivated to increase prices and protect their profitability. Formally we have

$$\pi = \alpha_{21} + \alpha_{22}\frac{l'}{l} + \alpha_{23}\left(\frac{w'}{w} - \frac{q'}{q}\right), \; 0 < \alpha_{2i} < 1 \quad (23)$$

Hence, besides the effect that the employment dynamics has on the rate of inflation, the reader may note that price inflation in (23) is positively related to nominal wage inflation, but it decreases with increments in labor productivity (lower labor costs).

Finally, in the case of the dynamics of labor intensity, we assume that workers will increase their intensity to work precisely when the rate of employment is low and, therefore, the relative size of the reserve army is high. As Marx suggested:

> "The development of the capitalist mode of production […] enables the capitalist, with the same outlay of variable capital, to set in action more labor by greater exploitation (extensive or intensive) of each individual labor power […] The overwork of the employed part of the working class swells the ranks of the reserve, whilst conversely the greater pressure that the latter by its competition exerts on the former, forces these to submit to overwork and to subjugation under the dictates of capital. The condemnation of one part of the working class to enforced idleness by the overwork of the other part, and the converse, becomes a means of enriching the individual capitalists, and accelerates at the same time the production of the industrial reserve army on a scale corresponding with the advance of social accumulation" (Marx [1867], 2010:629–30)

In addition, we also include the possibility that "in proportion as the use of machinery spreads, and the experience of a special class of workmen habituated to machinery accumulates, the rapidity and intensity of labor increases" (Marx [1867], 2010:412); however such effect cannot be unbounded but as the intensity of labor becomes higher, there is a pressure to reduce future labor intensity even because of physiological limits.[17] These intuitions are represented in expression (24), where labor intensity also increases with higher real wages; that is, higher wages boost employee morale and motivation, a hypothesis also sustained by the efficiency wage theories of the labor market, where workers are not seen as a hired input in much the same way as capital, and unlike capital can choose levels of effort.[18]

$$\frac{\epsilon'}{\epsilon} = \alpha_{31} - \alpha_{32}l + \alpha_{33}\frac{m'}{m} - \alpha_{34}\epsilon + \alpha_{35}\left(\frac{w'}{w} - \pi\right), \; 0 < \alpha_{3i} < 1 \quad (24)$$

In expression (24) $\alpha_{32}l$ deserves particular interest since it represents the power of the working class to reduce labor intensity at a given rate of employment; a power that also depends on the relative size of the reserve army of labor.

3. Interpreting some complex dynamics

Our dynamic system in its complete version can be captured by first combining (15), (16), (21), and (23) to get a reduced expression of the rate of growth of the employment rate (l'/l), as indicated in (25). Also, we combine (4), (7), (11), (22), and (23) to obtain expression (26) for the rate of change of the wage share (ω'/ω), while combining (4), (7), (11), and (24) gives expression (27) for the rate of change of labor intensity (ϵ'/ϵ), and renaming (9) and (28) give an expression for the rate of growth of mechanization (m'/m). As a result, the following system of equations is obtained.

$$\frac{l'}{l} = \left(\frac{1-\alpha_{23}}{1+\alpha_{22}-\alpha_{23}}\right)\left\{s\left[\frac{h(1-\omega)q - \delta m}{m+\rho h\omega q}\right] - \left\{n + \left(\frac{1}{1-\alpha_{23}}\right)\left(\alpha_{21} + \alpha_{23}\frac{\omega'}{\omega}\right) + \frac{\delta m + m' + \rho h[q\omega' + q'\omega]}{m+\rho h\omega q}\right\}\right\} \quad (25)$$

$$\frac{\omega'}{\omega} = \left[\frac{1-\alpha_{23}}{1-\alpha_{23}(1-\alpha_{13})}\right]\left[-\alpha_{11} + \alpha_{12}l - \frac{q'}{q} - \left(\frac{\alpha_{13}}{1-\alpha_{23}}\right)\left(\alpha_{21} + \alpha_{22}\frac{l'}{l}\right)\right] \quad (26)$$

$$\frac{\epsilon'}{\epsilon} = \alpha_{31} - \alpha_{32}l + \alpha_{33}\frac{m'}{m} - \alpha_{34}\epsilon + \alpha_{35}\left(\frac{\omega'}{\omega} + \frac{q'}{q}\right) \quad (27)$$

$$\frac{m'}{m} = \gamma_m \quad (28)$$

$$q = q(\epsilon, m) \quad (29)$$

Eqs. (25)–(29) form an autonomous non-linear and complex system capable to yield the dynamic behavior for *three state variables*: the rate of growth of the employment rate l, the wage share ω, and the intensity of labor ϵ. Here the parameters of the system are the saving rate s, the growth rate of the labor force n, the depreciation rate δ, the hours of work h, the growth rate of mechanization γ_m, and all the terms α_{ij}. Also, the system is capable to explain the dynamics of multiple (implicit) endogenous variables like the accumulation rate, the rate of profit, the rate of inflation, and others. This complex model is supposed to capture the essence of what we may call *an endogenous business cycle model of*

[17] "Machinery does not wear out exactly in the same ratio in which it is used. Man, on the contrary, decays in a greater ratio than would be visible from the mere numerical addition of work" (Marx [1865] 2010:141).

[18] A masterful survey, a lucid and systematic and yet critical account of the efficiency wages theories is found in Weiss (1991).

Marxist inspiration. To make sense of this complex model, some simplified versions can be considered.

3.1. Model A: an equivalent of Goodwin's (1967) model

We start our analysis by exploring a 2-D version of the system (25)–(29) built under the following assumptions:

- (i) All net profits go to the process of capital accumulation, i.e., $s = 1$
- (ii) The hours of work are normalized, i.e., $h = 1$
- (iii) Depreciation of constant capital is negligible, i.e., $\delta = 0$
- (iv) Wages are paid after production, i.e., $\rho = 0$
- (v) Labor productivity will only depend on mechanization and the intensity of labor and, for simplicity, it is specified as indicated in (30):

$$q(\epsilon, m) = \alpha_{40}\epsilon m, \ \ \alpha_{40} > 0 \tag{30}$$

- (vi) There is no inflation, i.e., $\alpha_{21} = \alpha_{22} = \alpha_{23} = 0, \pi = 0$
- (vii) Labor intensity remains constant, i.e., $\alpha_{31} = \alpha_{32} = \alpha_{33} = \alpha_{34} = \alpha_{35} = 0, \epsilon' = 0$.

When applying these assumptions to the (25)–(29) system, we obtain a simplified model represented by (31) and (32) that we name *model A* and where the endogenous state variables are the rate of employment l and the wage share ω.

$$\frac{l'}{l} = -(n+\gamma_m) + \alpha_{40}\epsilon(1-\omega) \tag{31}$$

$$\frac{\omega'}{\omega} = -(\alpha_{11}+\gamma_m) + \alpha_{12} l \tag{32}$$

Given assumptions (i) to (vii), γ_m is equivalent to the growth rate of labor productivity (q'/q), and α_{10} is equivalent to the inverse of the (constant) capital-output ratio (Q/A) if $\epsilon = 1$. Thus, Eqs. (31) and (32) are strictly equal to Eqs. (1) and (2) of Goodwin's (1967) growth cycle model. In other words, Goodwin's (1967) model may be considered as a particular case of the Marxian model represented by the system (25)–(29).

In model A, non-trivial equilibrium points corresponding to the steady-state $l' = \omega' = 0$ are given by:

$$l^* = \frac{\alpha_{11}+\gamma_m}{\alpha_{12}}, \ \ \omega^* = \frac{\alpha_{40}\epsilon - (n+\gamma_m)}{\alpha_{40}\epsilon} \tag{33}$$

This equilibrium has economic sense when $\omega^* < 1$, condition that is fulfilled when $\epsilon > (n+\gamma_m)/\alpha_{40}$. Also, it is required that $l^* < 1$, implying that $\alpha_{11} + \gamma_m < \alpha_{12}$. Given these conditions, the Jacobian of the system (31)-(32) evaluated at (l^*, ω^*) has the following trace and determinant:

$$\tau_A = 0, \ \ \Delta_A = (\alpha_{11}+\gamma_m)[\alpha_{40}\epsilon - (n+\gamma_m)] \rangle 0 \tag{34}$$

Thus, the system generates clockwise cyclical solutions in the $\omega - l$ space. We should note that the period (T) of these solutions decreases when labor intensity is higher,[19] as suggested by the numerical simulations represented in Fig. 1.[20]

3.2. Model B: Goodwin's (1967) model with (implicit) endogenous inflation

Here we analyze an extension of Goodwin's (1967) model by applying assumptions (i)-(v), (vii) on the system (25)–(29), and removing assumption (vi), so the price level is no more constant, but it moves according to an (implicit) endogenous rate of inflation given by expression (23) with $\alpha_{2i} > 0$ for all i. Given this change, and after some manipulations, we obtain a 2-D complex system represented by (35) and (36) which we name *model B*.

$$\frac{l'}{l} = \frac{[-(n+\gamma_m) + \alpha_{40}\epsilon(1-\omega)][1-\alpha_{23}(1-\alpha_{13})] + \gamma_m\alpha_{23} - [\alpha_{21} - \alpha_{23}(\alpha_{11} - \alpha_{12} l)]}{1+\alpha_{22} - \alpha_{23}(1-\alpha_{13})} \tag{35}$$

$$\frac{\omega'}{\omega} = \frac{[-(\alpha_{11}+\gamma_m) + \alpha_{12} l](1+\alpha_{22}-\alpha_{23}) - \alpha_{13}\{\alpha_{21} + \alpha_{22}[-(n+\gamma_m) + \alpha_{40}\epsilon(1-\omega)]\}}{1+\alpha_{22} - \alpha_{23}(1-\alpha_{13})} \tag{36}$$

In the steady-state, $l' = \omega' = 0$, non-trivial equilibrium points are:

$$l^* = \frac{(\alpha_{11}+\gamma_m)(1-\alpha_{23}) + \alpha_{13}\alpha_{21}}{\alpha_{12}(1-\alpha_{23})}, \ \ \omega^* = \frac{[\alpha_{40}\epsilon - (n+\gamma_m)](1-\alpha_{23}) - \alpha_{21}}{\alpha_{40}\epsilon(1-\alpha_{23})} \tag{37}$$

When $\alpha_{21} = \alpha_{22} = \alpha_{23} = 0$ the system (35) and (36) and the equilibrium point given by (37) are, respectively equal to expressions (31)–(33) corresponding to model A, that is, corresponding to Goodwin's (1967) model. In other words, model B represents the cyclical 2-D dynamics proposed by Goodwin (1967) but in an economy where inflation is influenced by the employment rate (demand-pull effect) and by the wage share (cost-push effect), as suggested by expression (23).

The trace and the determinant of the Jacobian of the system (35) and (36) evaluated at the equilibrium point (l^*, ω^*) are given by:

$$\tau_B = \frac{\alpha_{13}[\alpha_{22}\{(1-\alpha_{23})[\alpha_{40}\epsilon - (n+\gamma_m)] - \alpha_{21}\} - \alpha_{21}\alpha_{23}] - \alpha_{23}(1-\alpha_{23})(\alpha_{11}+\gamma_m)}{(1-\alpha_{23})[1+\alpha_{22}-\alpha_{23}(1-\alpha_{13})]} \tag{38}$$

$$\Delta_B = \frac{[\alpha_{13}\alpha_{21} + (\alpha_{11}+\gamma_m)(1-\alpha_{23})]\{(1-\alpha_{23})[\alpha_{40}\epsilon - (n+\gamma_m)] - \alpha_{21}\}}{(1-\alpha_{23})[1+\alpha_{22}-\alpha_{23}(1-\alpha_{13})]} \tag{39}$$

Under these conditions, and given the assumption that $0 < \alpha_{ij} < 1$ for all i, j, model B generates stable solutions when:

$$\tau_B \leq 0 \text{ if } \alpha_{13} \leq \alpha_{13}^B(\epsilon) = \frac{\alpha_{23}(1-\alpha_{23})(\alpha_{11}+\gamma_m)}{\alpha_{22}(1-\alpha_{23})[\alpha_{40}\epsilon - (n+\gamma_m)] - \alpha_{21}(\alpha_{22}+\alpha_{23})} \tag{40}$$

$$\Delta_B > 0 \text{ if } (1-\alpha_{23})[\alpha_{40}\epsilon - (n+\gamma_m)]\rangle\alpha_{21} \rightarrow \epsilon > \epsilon^B = \frac{\alpha_{21}}{\alpha_{40}(1-\alpha_{23})} + \frac{n+\gamma_m}{\alpha_{40}} \tag{41}$$

[19] The period of the ciclycal solutions is $T = \frac{2\pi}{\sqrt{(\alpha_{11}+\gamma_m)[\alpha_{40}\epsilon - (n+\gamma_m)]}}$.

[20] All the parameters and initial values used for numerical simulations have been chosen for illustrative purposes only.

Fig. 1. Effect of an exogenous change of labor intensity on cycles (model A).
Note: Simulations using parameters $n = 0.08,\ \gamma_m = 0,\ \alpha_{11} = 0.4,\ \alpha_{12} = 0.8,\ \alpha_{40} = 0.2$ and initial conditions $\omega_0 = 0.5, l_0 = 0.5$. For the trajectories of each state variable, see Fig. B.1 in Appendix B.

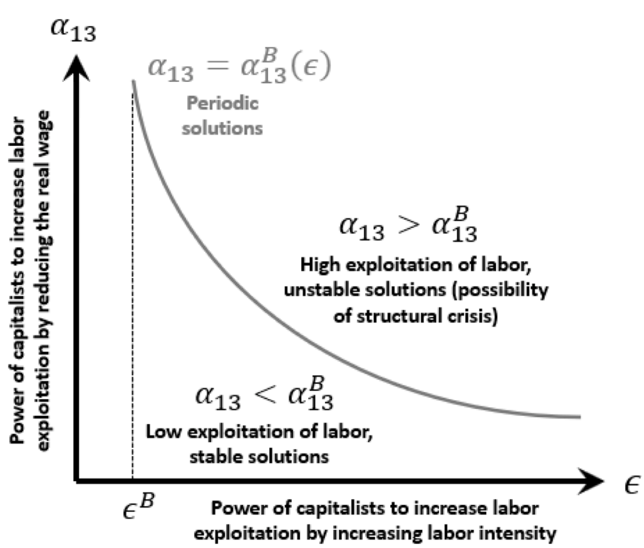

Fig. 2. Stability constraint for capitalist exploitation of labor (model B) (analytical).

With clockwise closed periodic solutions if $\alpha_{13} = \alpha_{13}^B$. In other words, given a sufficiently high intensity of labor ($\epsilon > \epsilon^B$), model B is locally stable when the term α_{13} is lower than α_{13}^B. Thus, when inflation is endogenous following the demand-pull and cost-push characterization presented in (23), economic stability requires that the power the capitalist class exerts to reduce the real wage below the objective of the working class – represented by α_{13} – does not exceed the upper bound given by α_{13}^B. Otherwise, if $\alpha_{13} > \alpha_{13}^B$ then *instability* emerges[21] with a possibility of *structural crisis:* a type of crisis that can be overcome only with an exogenous change in the structural parameters of the model and which is different from the endogenous and periodic crisis that emerge during the business cycle. If we also note that (40) describes an *inverse relationship* between α_{13}^B and the labor intensity ϵ, then it is possible to suggest that the capitalist exploitation of labor is constrained by the following condition: if the capitalist class increases its power to reduce the real wage below the target of the working class ($\uparrow\alpha_{13}$) then, ceteris paribus, labor intensity should decrease ($\downarrow\epsilon$) in order to sustain economic stability. Otherwise, *if a higher power of the capitalist class to reduce the real wage is not "compensated" by a lower labor intensity,* ceteris paribus, *then this higher exploitation of labor by capital makes the economy more vulnerable to instability and structural crisis* (until the structure of the model is changed to return to stability). Fig. 2 illustrates this result by presenting the inverse relationship between α_{13}^B and ϵ and the different

[21] The trace of the Jacobian matrix evaluated at the equilibrium point becomes positive.

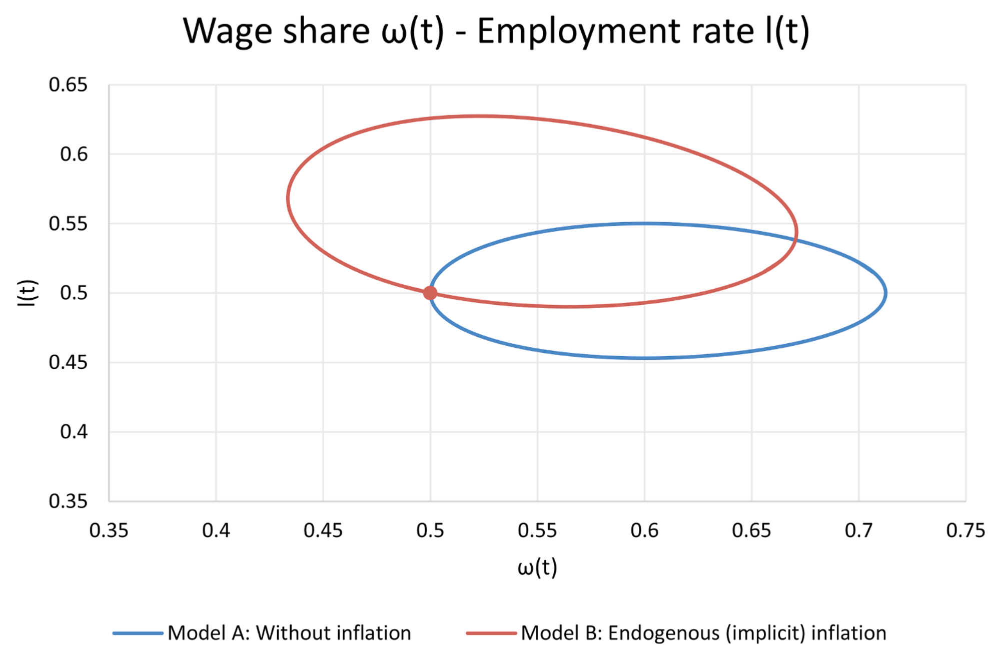

Fig. 3. Comparing periodic solutions between model A (without inflation) and model B (endogenous inflation).
Note: Simulations using parameters $n = 0.08,\ \gamma_m = 0,\ \alpha_{11} = 0.4,\ \alpha_{12} = 0.8,\ \alpha_{40} = 0.2, \alpha_{21} = 0.01, \alpha_{22} = 0.1, \alpha_{23} = 0.1, \alpha_{13} = 4.0909\ldots$ and initial conditions $\omega_0 = 0.5, l_0 = 0.5$. For the trajectories of each state variable, see Fig. B.2 in Appendix B.

conditions of stability: stable solutions when there is low exploitation of labor, periodic closed solutions when exploitation of labor fulfills expression (40), and unstable solutions that may cause a structural crisis when there is high exploitation of labor. In addition, Fig. 3 compares solutions of models A and B to emphasize the result that an endogenous rate of inflation can change the form, the location, and even the dynamics of the solutions of the system.[22]

3.3. *Model C: cycles with endogenous labor intensity and without inflation*

In previous 2-D models A and B, the labor intensity has been considered as an exogenous variable that may reduce the period of business cycles and even it can constraint the power of capitalists to increase labor exploitation. Now we present a 3-D model where prices remain constant while labor intensity is included as an *endogenous state variable*. More concretely, we use assumptions (i) to (vi) and remove assumption (vii), so labor intensity moves according to Eq. (27) (with $\alpha_{3i} > 0$ for all i). Also, for analytical simplicity, we include a new assumption:

- (viii) Mechanization is constant, i.e., $\gamma_m = 0$.

As a result, after some algebraic manipulations, we get a system represented by expressions (42)–(44) that we name *model C*.

$$\frac{l'}{l} = -n + \alpha_{40}\epsilon(1-\omega) \tag{42}$$

$$\frac{\omega'}{\omega} = -[\alpha_{31} + \alpha_{11}(1-\alpha_{35})] + [\alpha_{32} + \alpha_{12}(1-\alpha_{35})]l + \alpha_{34}\epsilon \tag{43}$$

$$\frac{\epsilon'}{\epsilon} = (\alpha_{31} - \alpha_{11}\alpha_{35}) - (\alpha_{32} - \alpha_{12}\alpha_{35})l - \alpha_{34}\epsilon \tag{44}$$

[22] For other implications of an endogenous inflation in Goodwin's model, see Flaschel (2010:393–96)

In the steady-state, $l' = \omega' = \epsilon' = 0$, non-trivial equilibrium points are:

$$l^* = \frac{\alpha_{11}}{\alpha_{12}},\ \omega^* = 1 - \frac{n\alpha_{12}\alpha_{34}}{\alpha_{40}(\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})},\ \epsilon^* = \frac{\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32}}{\alpha_{12}\alpha_{34}} \tag{45}$$

In analogy with the mathematical analysis presented by Dávila-Fernández and Sordi (2019), it can be proved that model C is locally stable at $(l^*, \omega^*, \epsilon^*)$ given the following condition (see Appendix A.1 for a mathematical proof):

$$\alpha_{12}\alpha_{35} < \alpha_{32} < \alpha_{12}\left(\frac{\alpha_{31}}{\alpha_{11}}\right) \tag{46}$$

Therefore, model C is locally stable when the *power* of workers to reduce capitalist exploitation through labor intensity at a given rate of employment – represented by α_{32} – is *bounded* by the limits given in (46). If that power falls out of those limits, then instability and structural crisis may emerge. Additionally, it is possible to apply the Hopf bifurcation theorem for 3D systems to prove that in the neighbourhood of the critical value:

$$\alpha_{32}^{HB} = \alpha_{12}\alpha_{35} \tag{47}$$

Model C has a *persistent cyclical behavior*, that is, the system presents *limit cycles* (see Appendix A.2 for a mathematical proof). Numerical simulations of model C suggest that the Hopf bifurcation identified in the neighborhood of α_{32}^{HB} is *supercritical* and associated with multiple *stable limit cycles*. Those limit cycles emerge depending on initial conditions, as Fig. 4 suggests using different initial values for labor intensity (ϵ_0) while other parameters and initial conditions remain constant. The simulations also suggest that the wage share and the employment rate present cyclical dynamics, and labor intensity may present stable monotone dynamics around the critical value α_{32}^{HB} (see Fig. B.3 in Appendix B). However, when model C is simulated using values of $\alpha_{32} \gg \alpha_{32}^{HB}$, it is possible to find stable spirals even for labor intensity solutions, emerging peculiar dynamics for the entire model (Fig. 5). Also, it is found that different initial conditions for the wage share and the employment rate may cause different limit cycles while keeping constant the initial value of the labor intensity (Fig. 6).

3.4. Model D: cycles with endogenous labor intensity and endogenous inflation

Finally, we analyze the complete system given by expressions (25)-(29) by using assumptions (i) to (v) to get a 3-D complex model where labor intensity is an endogenous variable and there is an (implicit) endogenous rate of inflation. We name this system as *model D*, and for simplicity, we represent it just in general terms as[23]:

$$\frac{l'}{l} = F_1(l,\omega,\epsilon),\ \ \frac{\omega'}{\omega} = F_2(l,\omega,\epsilon),\ \ \frac{\epsilon'}{\epsilon} = F_3(l,\omega,\epsilon) \tag{48}$$

In the steady-state, $l' = \omega' = \epsilon' = 0$, non-trivial equilibrium points are:

$$l^* = \frac{(\alpha_{11}+\gamma_m)(1-\alpha_{13})+\alpha_{13}\alpha_{21}}{\alpha_{12}(1-\alpha_{23})} \tag{48}$$

$$\omega^* = 1 - \frac{\alpha_{12}\alpha_{34}[\alpha_{21}+(n+\gamma_m)(1-\alpha_{23})]}{\alpha_{40}\{\alpha_{12}(1-\alpha_{23})[\alpha_{31}+(\alpha_{33}+\alpha_{35})\gamma_m]-\alpha_{32}[(\alpha_{11}+\gamma_m)(1-\alpha_{23})+\alpha_{13}\alpha_{21}]\}} \tag{49}$$

$$\epsilon^* = \frac{(1-\alpha_{23})\{(\alpha_{12}\alpha_{31}-\alpha_{11}\alpha_{32})+\gamma_m[\alpha_{12}(\alpha_{33}+\alpha_{35})-\alpha_{32}]\}-\alpha_{13}\alpha_{21}\alpha_{32}}{\alpha_{12}\alpha_{34}(1-\alpha_{23})} \tag{51}$$

A formal proof of the local stability of model D is left for future discussion. Even so, intuitions from models B and C and preliminary numerical simulations suggest that in model D the capitalist exploitation of labor is constrained by a condition analogous to the one presented in model B. Thus, based on the stability analysis and the Hopf bifurcation theorem for 3-D dynamical systems combined with numerical simulations, we identify that our simulations of model D are stable when $\alpha_{32} > \alpha_{32}^{HBD}$. Also, we find the existence of limit cycles in the neighbourhood of α_{32}^{HBD}: a critical value that can be expressed as a *function* of α_{13}, as suggested in (52).

$$\alpha_{32}^{HBD} = \alpha_{32}^{HBD}(\alpha_{13}) \tag{52}$$

Therefore, for each value of α_{13} it is necessary to adjust α_{32} according to (52) to sustain economic stability. We interpret this result in the following way: *for a given power of the capitalist class to reduce the real wage below the target of the working class (α_{13}), if there is a high exploitation of labor represented by a low power of the working class to reduce labor intensity ($\alpha_{32} \ll \alpha_{32}^{HBD}$), then this high exploitation of labor by capital makes the economy vulnerable to instability and structural crisis*.

Fig. 7 illustrates this result by presenting our multiple numerical simulations of model D where stable limit cycles are identified when, for each value of α_{13}, we identify the respective value α_{32}^{HBD} according to a numerical approximation of expression (52); this process is illustrated in Fig. 8 for numerical values that generate economically meaningful periodic solutions. At least for these simulations, it can be distinguished three situations analogous to those represented in Fig. 2 for model B: stable (spiral) solutions when there is low exploitation of labor, limit cycles when exploitation of labor is in the neighbourhood of expression (52), and unstable (spiral) solutions that may cause a structural crisis when there is high exploitation of labor. For the case of the stability constraint illustrated in Fig. 8, it is remarkable the non-monotonous numerical relationship we find between α_{13} and α_{32}^{HBD}, even with the possibility to identify a local maximum at α_{12}^*. The potential concavity of (52) may suggest a structural change in the power relationship between capitalists and workers during their struggle when defining wages and labor intensity: when $\alpha_{12} < \alpha_{12}^*$ the power of both workers (to reduce the intensity of labor) and capitalists (to reduce the real wage) should increase in order to obtain the limit cycles; however, when $\alpha_{12} > \alpha_{12}^*$ an *inverse relationship* emerges where stability seems to be sustained with the combination of an *increasing* power of the capitalists and a *decreasing* power of the working class.

Although these preliminary results obtained from our numerical simulations are not enough evidence for a robust conclusion, they bring some relevant insights about the contradiction between the capitalist power to reduce the real wage and the working-class power to reduce labor intensity when studying the cyclical dynamics of capitalism and its stability, at least in a Marxian context. In this sense, we emphasize the importance of a future mathematical and economical study of model D following a more rigorous approach in order to gain better intuitions on the interaction between *class-power and crisis.*

4. Conclusions

Although Marx placed a great deal of emphasis on the phenomenon of crisis or the periodical breakdown of capitalism, he was hardly less articulate in speaking of recurrent "industrial cycles" by which he meant what we have since become accustomed to call "the business cycle." But even accepting that Marx did not work a complete theory of business cycles, except in the most general and fragmented terms, such a theory can be reconstructed from his work. As Basu (2017) has correctly pointed out, at many places in his texts, Marx uses the term "crisis" to refer to what we would today call business cycle recessions, i.e. the downturn phases of regular business cycles. Marx saw crises as functional for capitalism, as corrective of underlying imbalances that are generated by the development of the system. The crisis removes obstacles to accumulation by imposing a new discipline on the working class, creating conditions in which workers have no choice, but to accept higher exploitation and insecurity. Thus, a plentiful supply of unemployed workers is setting the conditions to restore the profitability of the capitalist system and the consequent restoration of profitability allows business to expand rapidly without raising wages so long as there is a large reserve of unemployed (or underemployed) workers. A phase of growing animation and prosperity will in turn follow the crisis and in this way the capitalist system will show a cyclical interaction of accumulation, employment, exploitation and income distribution between workers and capitalists. In our modern terminology the system will exhibit cycles of booms and slumps that neither have a fixed and regular character, nor an exact length or amplitude.

Though there are modern efforts in the literature to reconstruct and fomalize Marx's theory of business cycle, either do not entirely rely on Marx's great insights on the structure and dynamics of production in capitalist economies or often leave out important aspects closely linked to the macro-dynamics contained in Marx. The most known and elegant

[23] To study the dynamics of model D we built a notebook in *Wolfram Mathematica* available as supplementary material. More details are available upon request to the authors

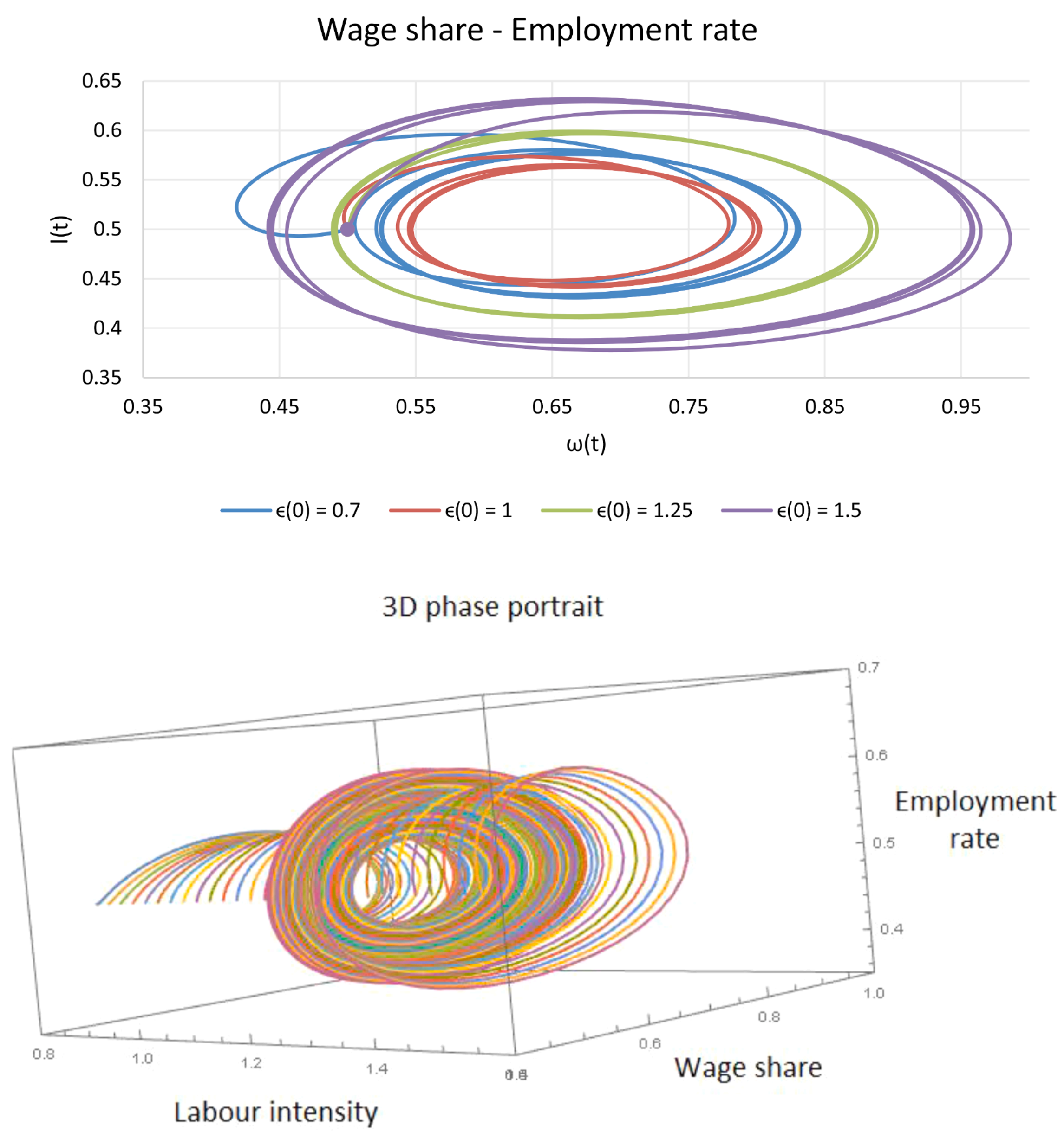

Fig. 4. Limit cycles using different initial conditions for the labor intensity (model C).
Note: Simulations using parameters $n = 0.08,\ \gamma_m = 0,\ \alpha_{11} = 0.4,\ \alpha_{12} = 0.8,\ \alpha_{40} = 0.2, \alpha_{31} = 0.1, \alpha_{32} = 0.08001, \alpha_{33} = 0.1, \alpha_{34} = 0.05, \alpha_{35} = 0.1$ and initial conditions $\omega_0 = 0.5, l_0 = 0.5$. For the trajectories of each state variable, see Fig. B.3 in Appendix B.

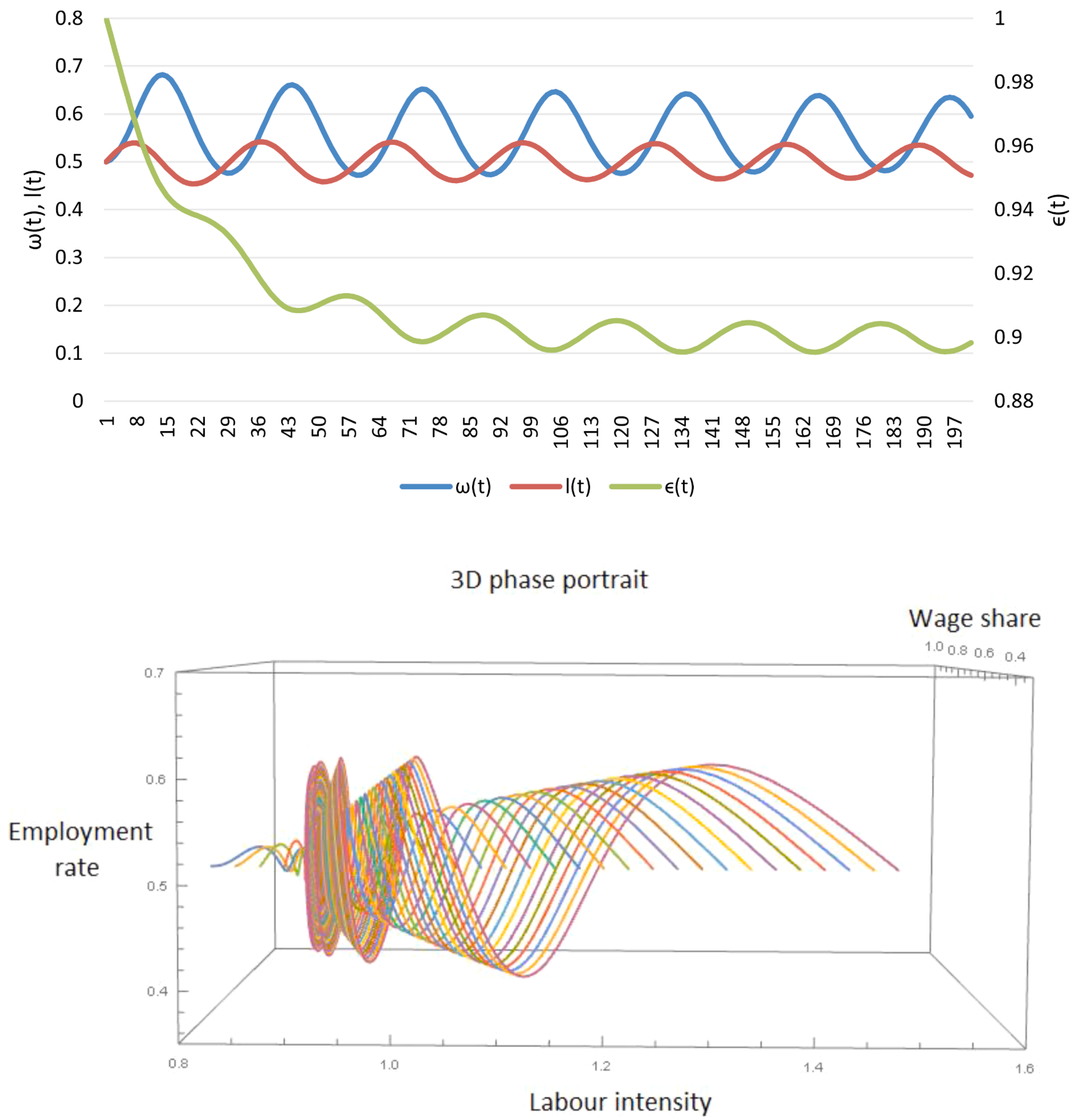

Fig. 5. Example of solutions with stable spirals for the trajectory of the labor intensity (model C).
Note: Simulations using parameters $n = 0.08,\ \gamma_m = 0,\ \alpha_{11} = 0.4,\ \alpha_{12} = 0.8,\ \alpha_{40} = 0.2, \alpha_{31} = 0.1, \alpha_{32} = 0.08001 + 0.03, \alpha_{33} = 0.1, \alpha_{34} = 0.05, \alpha_{35} = 0.1$ and multiple initial conditions.

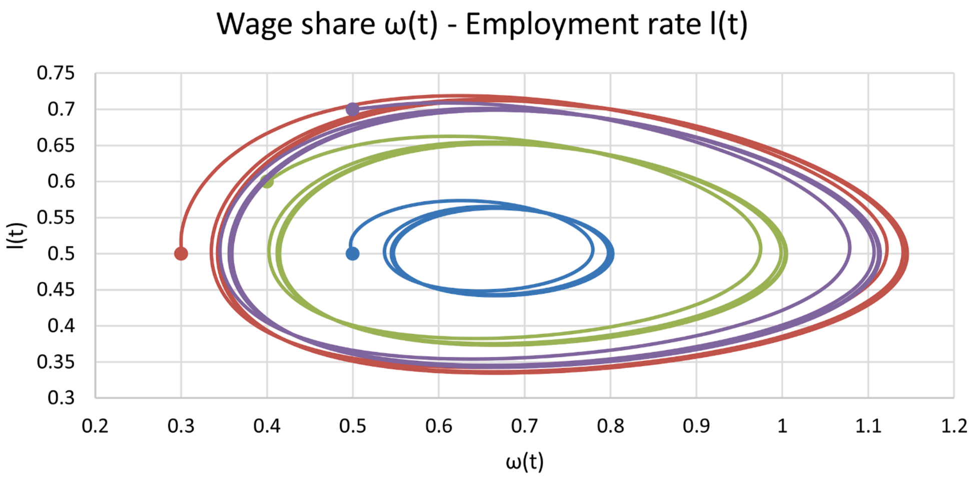

Fig. 6. Limit cycles using different initial conditions for the wage share and the employment rate (model C)
Note: Simulations using parameters $n = 0.08,\ \gamma_m = 0,\ \alpha_{11} = 0.4,\ \alpha_{12} = 0.8,\ \alpha_{40} = 0.2, \alpha_{31} = 0.1, \alpha_{32} = 0.08001, \alpha_{33} = 0.1, \alpha_{34} = 0.05, \alpha_{35} = 0.1$ and initial condition $\epsilon_0 = 1$..

dynamic formalization of Marx's view of distributive cycles presented by Goodwin (1967) though recognizes the key role that the reserve army of labor plays in the course of profitability and capital accumulation, misses from the analysis an explicit account of a very valuable insight of Marx: that labor productivity is endogenous and that it changes according to changes in labor-saving machinery, real wages as well as by the intensity of labor.

The modeling structure of the endogenous business cycle that we offer adds up to other efforts by integrating key elements of marxist inspiration: (a) the possibility of endogenous labor productivity, (b) the existence of exploitation through changes in labor intensity, and (c) the underlying conflict between labor and capital in the price-wage setting process. The model forms an autonomous non-linear and complex system that is capable to yield the dynamic behavior for *three state variables*: the rate of growth of the employment rate l, the wage share ω, and the intensity of labor ϵ. It is also general enough to admit, as solutions, several particular cases.

In a first 2-D version of the system where labor intensity is constant, labor productivity grows at a uniform rate, and there is no price inflation, the system reproduces exactly Goodwin's growth cycle model. Interesting enough, an increase in the parameter that represents labor intensity will bring about shorter cycles.

By removing the fixed price assumption, a second simplified version of the general model allow us to identify a unique singular point that presents local stability. The analysis of the trace and the determinant of this 2D complex system shows that when inflation is considered as an (implicit) endogenous variable, economic stability requires that the parameter that represents the power that businesses exert over the real wage be below some upper bound working-class target. Notable, such upper bound has an inverse relationship with labor intensity, thus, we show that a sufficiently high power of the capitalist class to reduce the real wage combined with a sufficiently high labor intensity will make the economy more vulnerable to instability and a structural crisis. This type of crisis is different from the endogenous and periodic crisis that emerge during the business cycle and can only be overcome with an exogenous change in the parameters or the structure of the model.

When labor intensity is included as an endogenous state variable and the price level is kept constant we obtain a 3D dynamical system (model C). The Hopf bifurcation theorem for 3D dynamical systems indicates that economic stability in this model requires that the power of workers to reduce capitalist exploitation through labor intensity (the "curvature coefficient") should be bounded by specific limits. Also, we provide an analytical demonstration and numerical simulations to prove the existence of multiple stable limit cycles depending on the initial conditions of the state variables; those limit cycles emerge when the power of workers to reduce exploitation through labor intensity falls in the neighborhood of a critical value that seems to generate supercritical bifurcations.

We present a most complete version of the general model in which both labor intensity and inflation are endogenous. With these assumptions, we obtain a more complex 3D dynamical system (model D) that is analyzed through the application of the Hopf theorem to numerical simulations of the model. Even though the bifurcation, in general, leads to stable limit cycles oscillations, we show that a sufficiently high power of the capitalist class to reduce the real wage combined with a sufficiently high power of the capitalist class to increase labor intensity will make the economy more vulnerable to instability and a structural crisis. In sum, in our most complete version of the general model a high enough exploitation of labor by capital makes the economy vulnerable to instability and structural crisis. In this sense, we estimate numerically the relationship between these types of power that is necessary for maintaining limit cycles, with the preliminary result that the relationship may be represented by a non-linear function that deserves a deeper discussion when studying the interaction between class-power and crisis. Such a discussion may provide an alternative interpretation of economic cycles and the role of power relations during each stage of the cycle, and it may be complemented by other recent interpretations like Mariolis et al. (2021), Nikiforos (2022), and others that identify different growth regimes within the cycle considering both supply and demand perspectives.

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CRediT authorship contribution statement

John Cajas Guijarro: Conceptualization, Methodology, Investigation, Formal analysis, Writing – original draft, Software, Visualization. **Leonardo Vera:** Conceptualization, Methodology, Investigation, Formal analysis, Writing – original draft.

Declarations of Competing Interest

None.

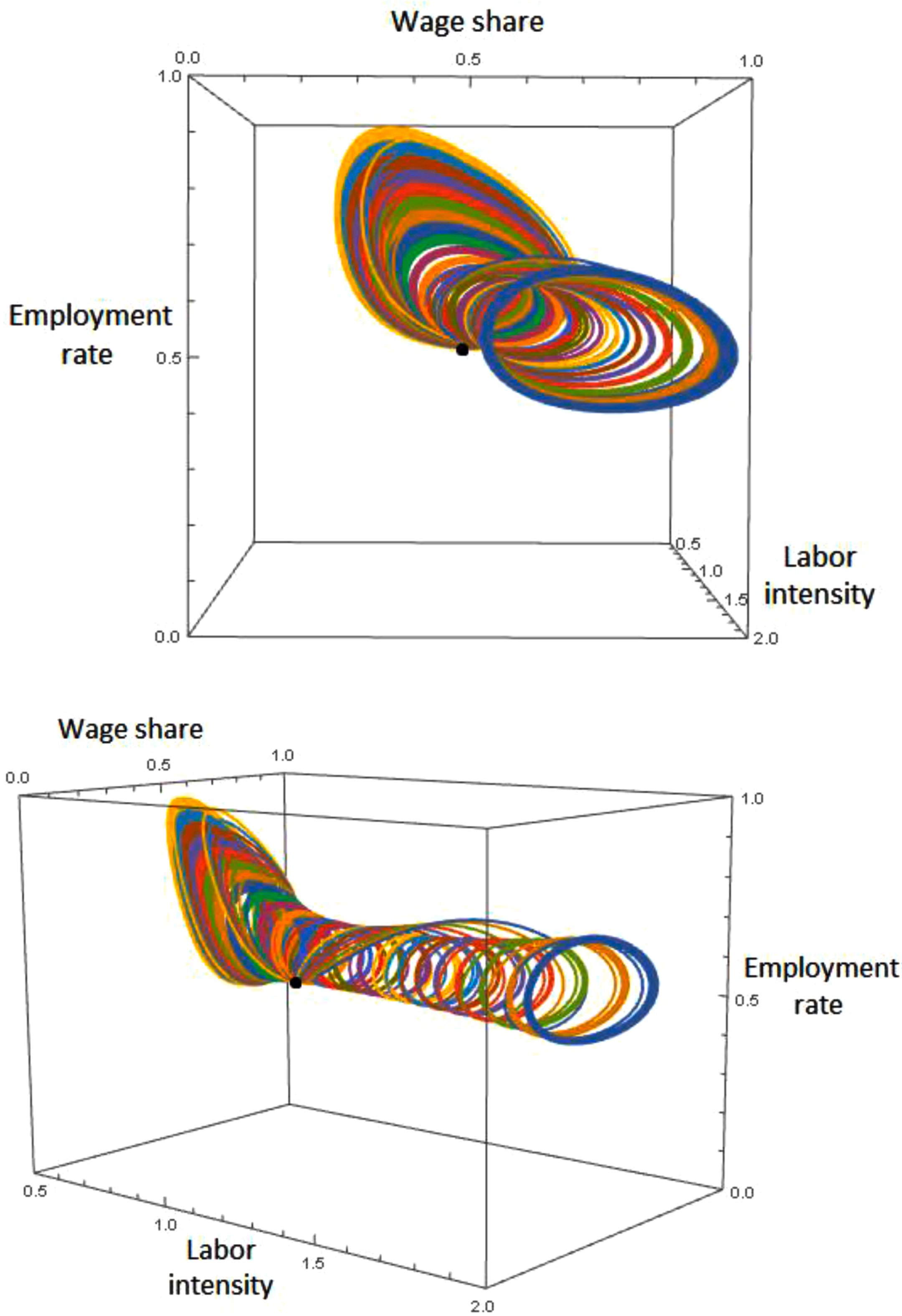

Fig. 7. Simulation of multiple limit cycles when $\boldsymbol{\alpha}_{13}$ and $\boldsymbol{\alpha}_{32}$ are changed following the stability constraint (model D).
Note: Simulations using parameters $n = 0.08$, $\gamma_m = 0$, $\alpha_{11} = 0.4$, $\alpha_{12} = 0.8$, $\alpha_{40} = 0.2, \alpha_{21} = 0.01, \alpha_{22} = 0.1, \alpha_{23} = 0.1, \alpha_{31} = 0.1, \alpha_{33} = 0.1, \alpha_{34} = 0.05, \alpha_{35} = 0.1$, with α_{13}, α_{32} given by the "stability constraint" presented in Fig. 8, and initial conditions $\omega_0 = 0.5, l_0 = 0.5, \epsilon_0 = 1$ (black point).

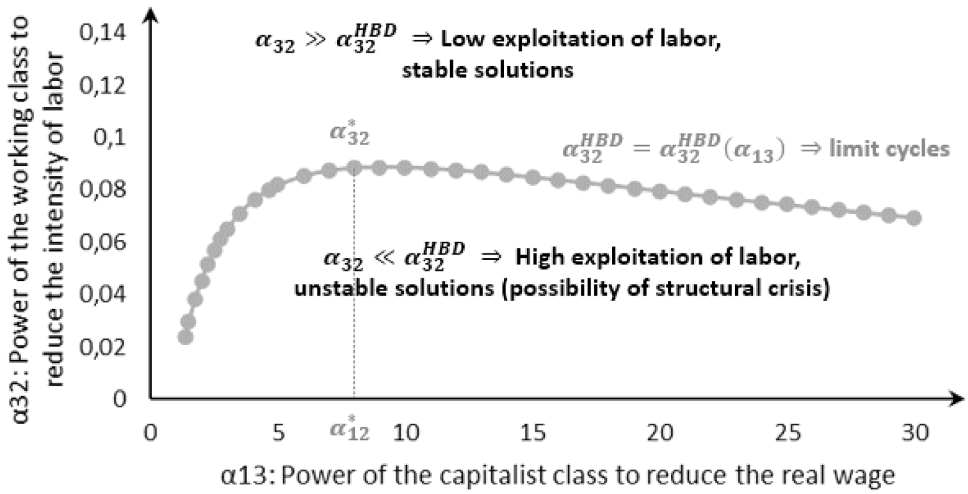

Fig. 8. Stability constraint for capitalist exploitation of labor (model D) (numerical simulation).
Note: Simulations using parameters $n = 0.08$, $\gamma_m = 0$, $\alpha_{11} = 0.4$, $\alpha_{12} = 0.8$, $\alpha_{40} = 0.2, \alpha_{21} = 0.01, \alpha_{22} = 0.1, \alpha_{23} = 0.1, \alpha_{31} = 0.1, \alpha_{33} = 0.1, \alpha_{34} = 0.05, \alpha_{35} = 0.1$, with α_{13}, α_{32} given by the "stability constraint", and initial conditions $\omega_0 = 0.5, l_0 = 0.5, \epsilon_0 = 1$.

Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.strueco.2022.08.002.

Appendix A

A.1. Proof of local stability (model C)

Following Dávila-Fernández and Sordi (2019), to analyze the local stability of the system (42)–(44) around the point $(l^*, \omega^*, \epsilon^*)$ given by (45), we linearize the model, obtaining:

$$\begin{bmatrix} l' \\ \omega' \\ \epsilon' \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} \begin{bmatrix} l - l^* \\ \omega - \omega^* \\ \epsilon - \epsilon^* \end{bmatrix}$$

Where the elements J_{ij} of the Jacobian matrix evaluated at (l^*, ω^*, p^*) are:

$$J_{11} = \frac{\partial f_1(l,\omega,\epsilon)}{\partial l}\Big|_{(l^*,\omega^*,\epsilon^*)} = 0$$

$$J_{12} = \frac{\partial f_1(l,\omega,\epsilon)}{\partial \omega}\Big|_{(l^*,\omega^*,\epsilon^*)} = -\frac{\alpha_{11}(\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})\alpha_{40}}{\alpha_{12}^2\alpha_{34}}$$

$$J_{13} = \frac{\partial f_1(l,\omega,\epsilon)}{\partial \epsilon}\Big|_{(l^*,\omega^*,\epsilon^*)} = \frac{n\alpha_{11}\alpha_{34}}{\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32}}$$

$$J_{21} = \frac{\partial f_2(l,\omega,\epsilon)}{\partial l}\Big|_{(l^*,\omega^*,\epsilon^*)} = [\alpha_{32} + \alpha_{12}(1 - \alpha_{35})]\left[1 - \frac{n\alpha_{12}\alpha_{34}}{\alpha_{40}(\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})}\right]$$

$$J_{22} = \frac{\partial f_2(l,\omega,\epsilon)}{\partial \omega}\Big|_{(l^*,\omega^*,\epsilon^*)} = 0$$

$$J_{23} = \frac{\partial f_2(l,\omega,\epsilon)}{\partial \epsilon}\Big|_{(l^*,\omega^*,\epsilon^*)} = \alpha_{34}\left[1 - \frac{n\alpha_{12}\alpha_{34}}{\alpha_{40}(\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})}\right]$$

$$J_{31} = \frac{\partial f_3(l,\omega,\epsilon)}{\partial l}\Big|_{(l^*,\omega^*,\epsilon^*)} = \frac{(\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})(\alpha_{12}\alpha_{35} - \alpha_{32})}{\alpha_{12}\alpha_{34}}$$

$$J_{32} = \frac{\partial f_3(l,\omega,\epsilon)}{\partial \omega}\Big|_{(l^*,\omega^*,\epsilon^*)} = 0$$

$$J_{33} = \frac{\partial f_3(l,\omega,\epsilon)}{\partial \epsilon}\Big|_{(l^*,\omega^*,\epsilon^*)} = -\alpha_{31} + \frac{\alpha_{11}\alpha_{32}}{\alpha_{12}}$$

Thus, the characteristic equation of the Jacobian matrix is:

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$$

Where:

$$b_1 = -\tau = \frac{\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32}}{\alpha_{12}}$$

$$b_2 = \begin{vmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{13} \\ J_{31} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix} = \alpha_{11}\left\{\frac{(\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})[\alpha_{32} + \alpha_{12}(1-\alpha_{35})]\alpha_{40}}{\alpha_{12}^2\alpha_{34}} - n\right\}$$

$$b_3 = -\Delta = \frac{\alpha_{11}(\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})[(\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})\alpha_{40} - n\alpha_{12}\alpha_{34}]}{\alpha_{12}^2\alpha_{34}}$$

$$b_1b_2 - b_3 = \frac{\alpha_{11}(\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})^2(\alpha_{32} - \alpha_{12}\alpha_{35})\alpha_{40}}{\alpha_{12}^3\alpha_{34}}$$

The necessary and sufficient condition for the local stability of the system at $(l^*,\omega^*,\epsilon^*)$ is that all the roots λ have negative real parts, something that simultaneously requires $b_1 > 0, b_2 > 0, b_3 > 0$ and $b_1b_2 - b_3 > 0$. If it is assumed that n is sufficiently low, these conditions are satisfied when $\alpha_{11}\alpha_{32} < \alpha_{12}\alpha_{31}$ and $\alpha_{32} > \alpha_{12}\alpha_{35}$. In other words, model C is stable if:

$$\alpha_{12}\alpha_{35} < \alpha_{32} < \alpha_{12}\left(\frac{\alpha_{31}}{\alpha_{11}}\right)$$

A.2. Proof of the existence of a Hopf bifurcation (model C)

Following the procedure presented by Dávila-Fernández and Sordi (2019), based on Gandolfo (2009), it is possible to use the Hopf bifurcation theorem for 3-D dynamical systems by taking α_{13} as a bifurcation parameter to prove that the system (42)-(44) has a family of periodic solutions that take the form of *limit cycles*. This proof requires two conditions: (HB1) the characteristic equation has a pair of purely imaginary eigenvalues and one eigenvalue with non-zero real part at the critical point α_{32}^{HB}; (HB2) the derivative of the real part of the complex eigenvalues of the characteristic equation with respect to α_{32} is different from zero at the critical value α_{33}^{HB}.

For condition (HB1), from Appendix A.1 we know that if n is sufficiently low, b_1, b_2, b_3 are positive when $\alpha_{32} < \alpha_{12}\left(\frac{\alpha_{31}}{\alpha_{11}}\right)$. Under these conditions, a Hopf bifurcation requires $b_1b_2 - b_3 = 0$, something that occurs when $\alpha_{32} = \alpha_{32}^{HB} = \alpha_{12}\alpha_{35}$. Therefore, (HB1) is satisfied when:

$$\alpha_{32} = \alpha_{32}^{HB} = \alpha_{12}\alpha_{35} \text{ and } \alpha_{11}\alpha_{32} < \alpha_{12}\alpha_{31}$$

Condition equivalent to:

$$\alpha_{32}^{HB} = \alpha_{12}\alpha_{35} \text{ and } \alpha_{35} < \frac{\alpha_{31}}{\alpha_{11}}$$

Next, to prove (HB2) we can obtain the derivatives of b_1, b_2, b_3 with respect to α_{32}:

$$\frac{\partial b_1}{\partial \alpha_{32}} = -\frac{\alpha_{11}}{\alpha_{12}}$$

$$\frac{\partial b_2}{\partial \alpha_{32}} = \frac{\alpha_{11}\alpha_{40}\{(\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32}) - \alpha_{11}[\alpha_{32} + \alpha_{12}(1-\alpha_{35})]\}}{\alpha_{12}^2\alpha_{34}}$$

$$\frac{\partial b_3}{\partial \alpha_{32}} = \frac{\alpha_{11}^2[n\alpha_{12}\alpha_{34} - 2\alpha_{40}(\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})]}{\alpha_{12}^2\alpha_{34}}$$

When $\alpha_{32} = \alpha_{32}^{HB}$ from (HB1) we know the characteristic equation has one real negative root $\lambda_1 < 0$ and two complex roots $\lambda_{2,3} = a \pm bi$ with $a = 0$. Under these conditions, it can be proved (see Appendix A.3 in Dávila-Fernández and Sordi, 2019 for a detailed mathematical deduction) that:

$$\frac{\partial b_1}{\partial \alpha_{32}} = -\frac{\partial \lambda_1}{\partial \alpha_{32}} - 2\frac{\partial a}{\partial \alpha_{32}}$$

$$\frac{\partial b_2}{\partial \alpha_{32}} = 2\lambda_1\frac{\partial a}{\partial \alpha_{32}} + 2b\frac{\partial b}{\partial \alpha_{32}}$$

$$\frac{\partial b_3}{\partial \alpha_{32}} = -b^2 \frac{\partial \lambda_1}{\partial \alpha_{32}} - 2\lambda_1 b \frac{\partial b}{\partial \alpha_{32}}$$

Thus, we get the following system of equations:

$$-X - 2Y = -\frac{\alpha_{11}}{\alpha_{12}}$$

$$2\lambda_1 Y + 2bZ = \frac{\alpha_{11}\alpha_{40}\{(\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32}) - \alpha_{11}[\alpha_{32} + \alpha_{12}(1 - \alpha_{35})]\}}{\alpha_{12}^2 \alpha_{34}}$$

$$-b^2 X - 2\lambda_1 bZ = \frac{\alpha_{11}^2 [n\alpha_{12}\alpha_{34} - 2\alpha_{40}(\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})]}{\alpha_{12}^2 \alpha_{34}}$$

Where:

$$X = \frac{\partial \lambda_1}{\partial \alpha_{32}}, Y = \frac{\partial a}{\partial \alpha_{32}}, Z = \frac{\partial b}{\partial \alpha_{32}}$$

When evaluating the solution of this system at α_{32}^{HB} for the derivative of the real part of the complex eigenvalues of the characteristic equation $\frac{\partial a}{\partial \alpha_{32}}$, we get:

$$\frac{\partial a}{\partial \alpha_{32}}\Big|_{\alpha_{32}^{HB}} = \frac{\alpha_{11}\{\alpha_{34}(b^2 + n\alpha_{11}) + \alpha_{40}[2\alpha_{11}(\alpha_{11}\alpha_{35} - \alpha_{31}) - \lambda_1(\alpha_{11} + \alpha_{11}\alpha_{35} - \alpha_{31})]\}}{2\alpha_{12}\alpha_{34}(b^2 + \lambda_1^2)}$$

Previous conditions guarantee that $\frac{\partial a}{\partial \alpha_{32}}\big|_{\alpha_{32}^{HB}} \neq 0$, then (HB2) is satisfied. Therefore, it is confirmed the existence of limit cycles for model B in the neighborhood of α_{32}^{HB}.

Appendix B

Figs. B.1–B.3

Fig. B.1. Trajectories of the state variables corresponding to Fig. 1 (model A).
Note: Simulations using parameters $n = 0.08$, $\gamma_m = 0$, $\alpha_{11} = 0.4$, $\alpha_{12} = 0.8$, $\alpha_{40} = 0.2$ and initial conditions $\omega_0 = 0.5, l_0 = 0.5$.

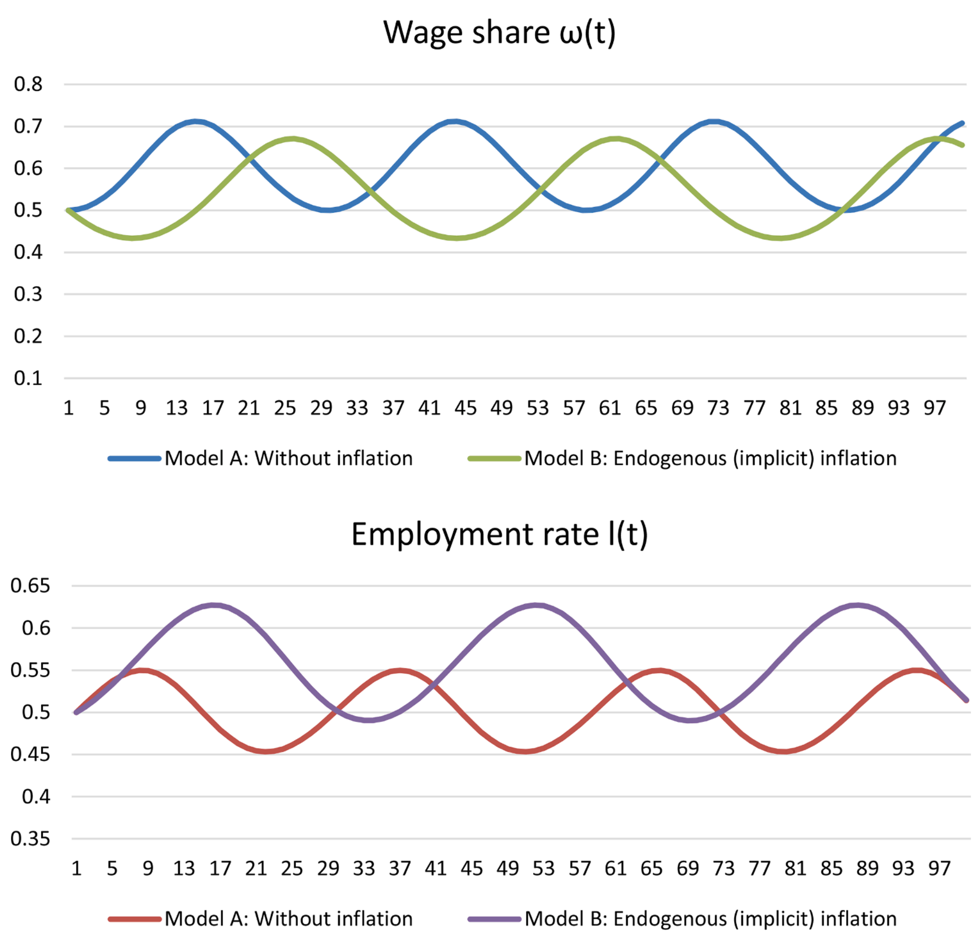

Fig. B.2. Trajectories of the state variables corresponding to Fig. 3 (comparison of models A and B).
Note: Simulations using parameters $n = 0.08,\ \gamma_m = 0,\ \alpha_{11} = 0.4,\ \alpha_{12} = 0.8,\ \alpha_{40} = 0.2, \alpha_{21} = 0.01, \alpha_{22} = 0.1, \alpha_{23} = 0.1, \alpha_{13} = 4.0909\ldots$ and initial conditions $\omega_0 = 0.5, l_0 = 0.5$.

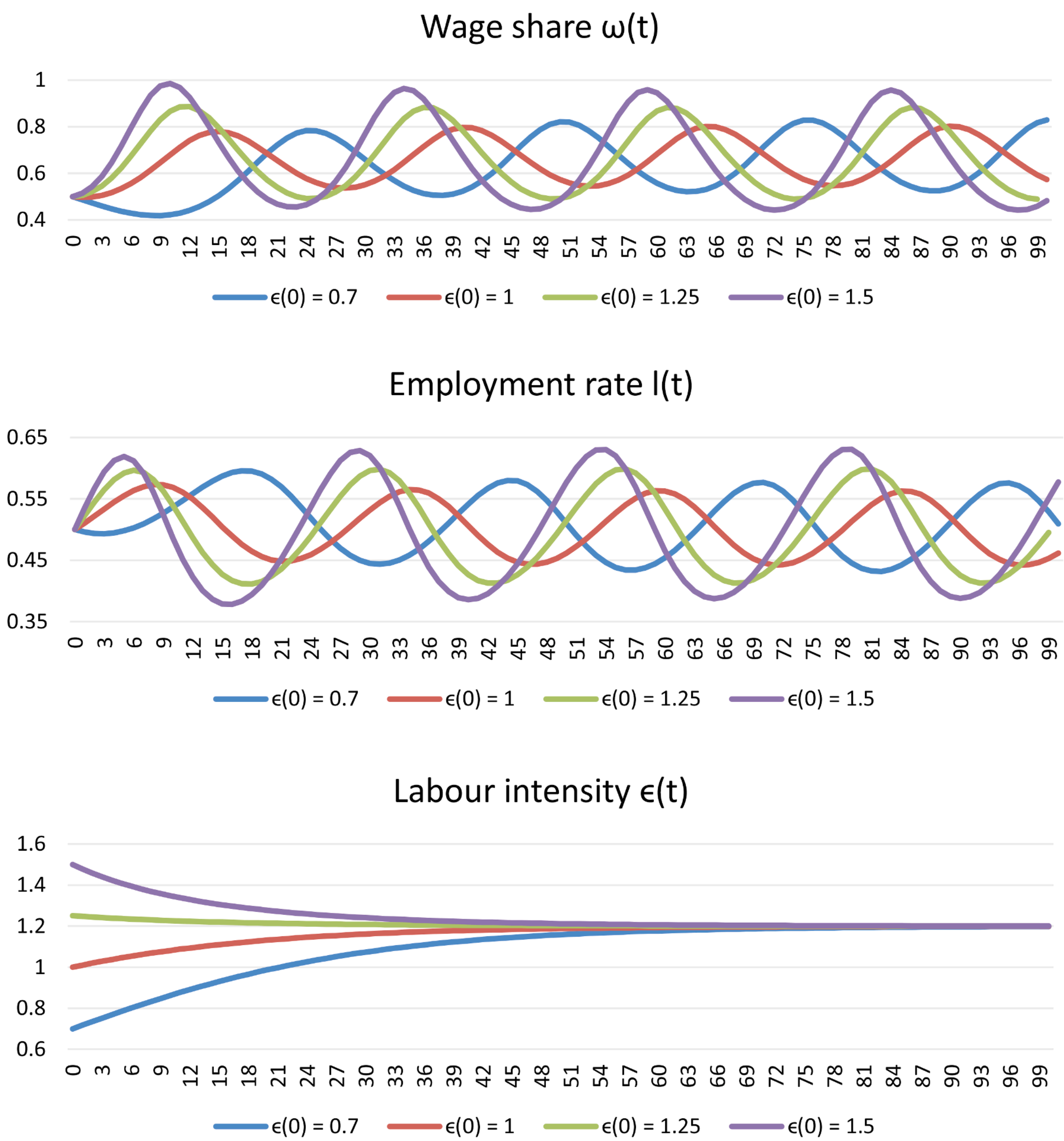

Fig. B.3. Trajectories of the state variables corresponding to Fig. 4 (model C).
Note: Simulations using parameters $n = 0.08,\ \gamma_m = 0,\ \alpha_{11} = 0.4,\ \alpha_{12} = 0.8,\ \alpha_{40} = 0.2, \alpha_{31} = 0.1, \alpha_{32} = 0.08001, \alpha_{33} = 0.1, \alpha_{34} = 0.05, \alpha_{35} = 0.1$ and initial conditions $\omega_0 = 0.5, l_0 = 0.5$.

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